MECHANICS OF SOLIDS (ME F211)

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Chapter-4

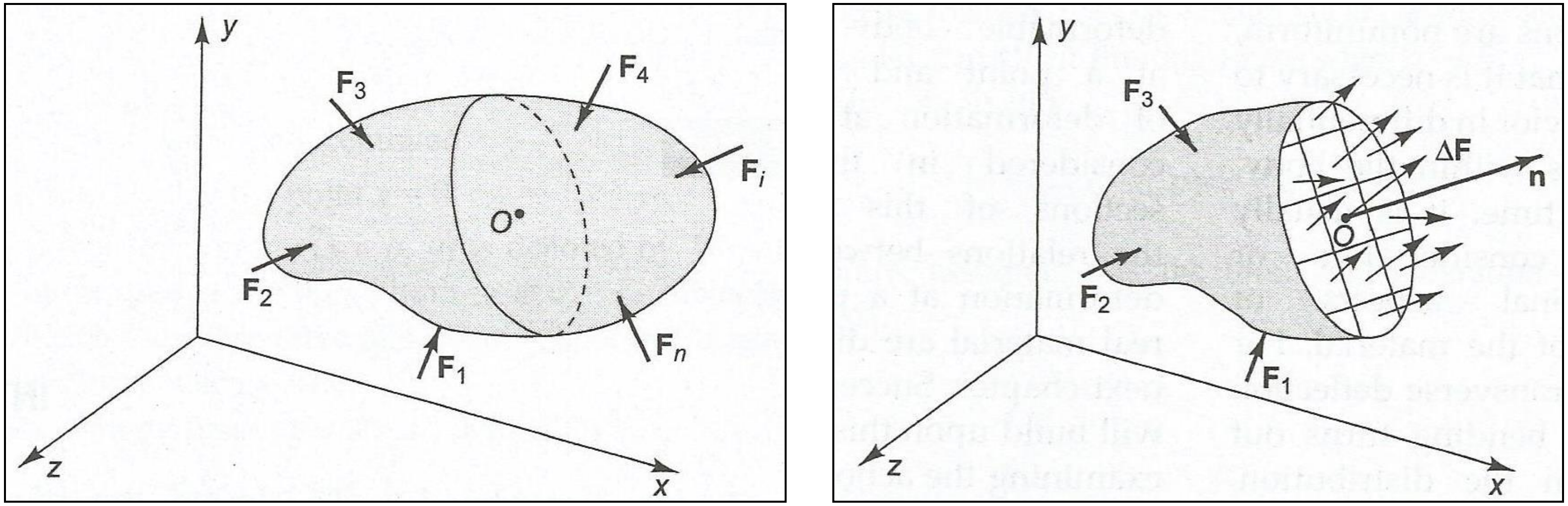
Stress and Strain

Objectives

- Stress
  - To know state of stress at a point
  - To solve for plane stress condition applications
- Strain
  - To know state of strain at a point
  - To solve for plane strain condition applications

Stress and Strain

Stress



Stress vector can be defined as

$$\overset{(n)}{T} = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

T is force intensity or stress acting on a plane whose normal is '**n**' at the point *O.*

Characteristics of stress

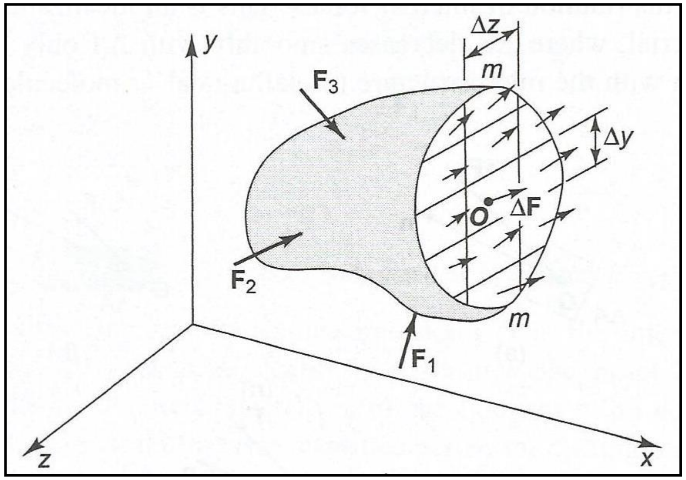
- The physical dimensions of stress are force per unit area.
- Stress is defined at a point upon an imaginary plane, which divides the element or material into two parts.
- Stress is a vector equivalent to the action of one part of the material upon another.
- The direction of the stress vector is not restricted

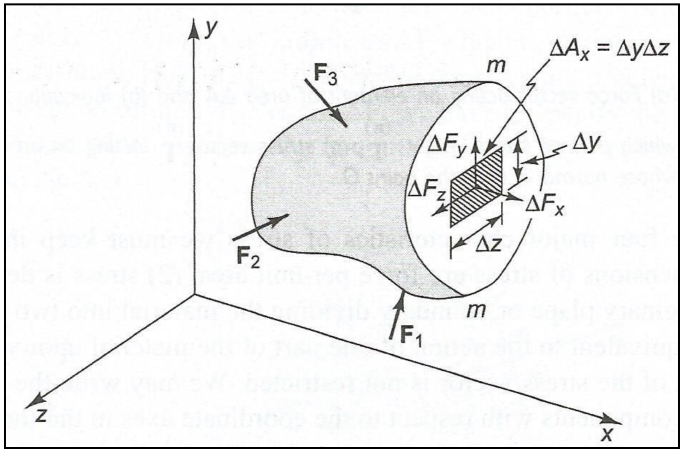
Stress vector may be written in terms of its components with respect to the coordinate axes in the form

$$\boxed{\overset{(n)}{T} = \overset{(n)}{T}_x\, i + \overset{(n)}{T}_y\, j + \overset{(n)}{T}_z\, k}$$

Stress and Strain

- Body cut by a plane '*mm*' passing through point '*O*' and parallel to y-z plane and
- Consider the free body of the left part of the plane '*mm*'
- Divide plane *mm* in large number of small areas, i.e. $\Delta y \times \Delta z$
- A force ΔF is acting on the small area ΔA (ΔA is centered on the point *O*).
- ΔF is inclined to the surface *mm* at some arbitrary angle.
- Figure shows the rectangular components of the force vector ΔF acting on the small area centered on point *O.*





Stress and Strain

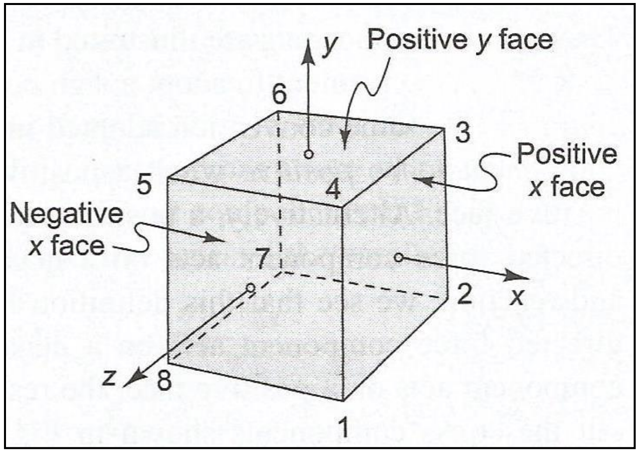
Definition of positive and negative faces

Positive face of given section

If the outward normal points in a positive coordinate direction then that face is called as positive face

Negative face of given section

If the outward normal points in a negative coordinate direction then that face is called as Negative face



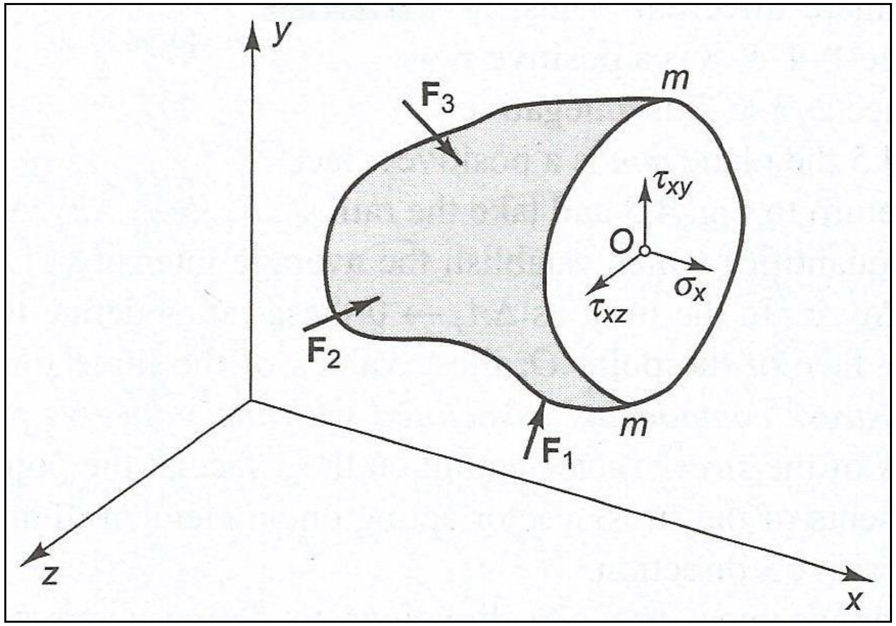
Face	Positive	Negative
x face	1-2-3-4	5-6-7-8
y face	3-4-5-6	1-2-7-8
z face	1-4-5-8	2-3-6-7

Stress and Strain

Stress components on positive *x* face

Normal Stress
$$\sigma_x = \lim_{\Delta A_x \to 0} \frac{\Delta F_x}{\Delta A_x}$$

Shear Stress
$$\tau_{xy} = \lim_{\Delta A_x \to 0} \frac{\Delta F_y}{\Delta A_x}$$
$$\tau_{xz} = \lim_{\Delta A_x \to 0} \frac{\Delta F_z}{\Delta A_x}$$



Similarly on *y* face σ_y, τ_{yx} & τ_{yz} and on *z* face σ_z, τ_{zx} & τ_{zy} stress components will exist.

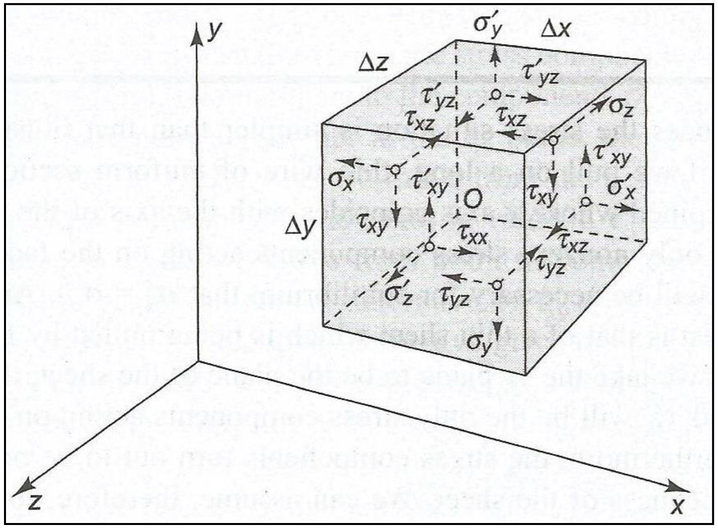
Stress and Strain

3-Dimentional State of Stress OR Triaxial State of Stress

A knowledge of the nine stress components is necessary in order to determine the components of the stress vector *T* acting on an arbitrary plane with normal **n**.

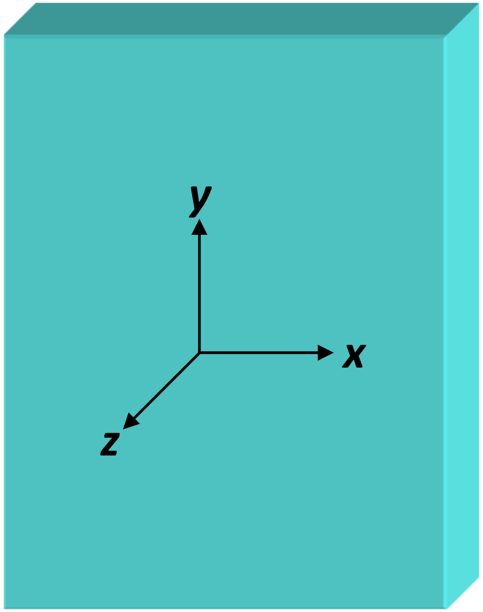
$$
\begin{matrix}
\sigma_x & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z
\end{matrix}
$$

Stress components acting on the six sides of a parallelepiped.



Plane stress condition

- Stress components in the *z* direction has a very small value compared to the other two directions and moreover they do not vary throughout the thickness.
- For example: Thin sheet which is being pulled by forces in the plane of the sheet.
- The state of stress at a given point will only depend upon the four stress components.

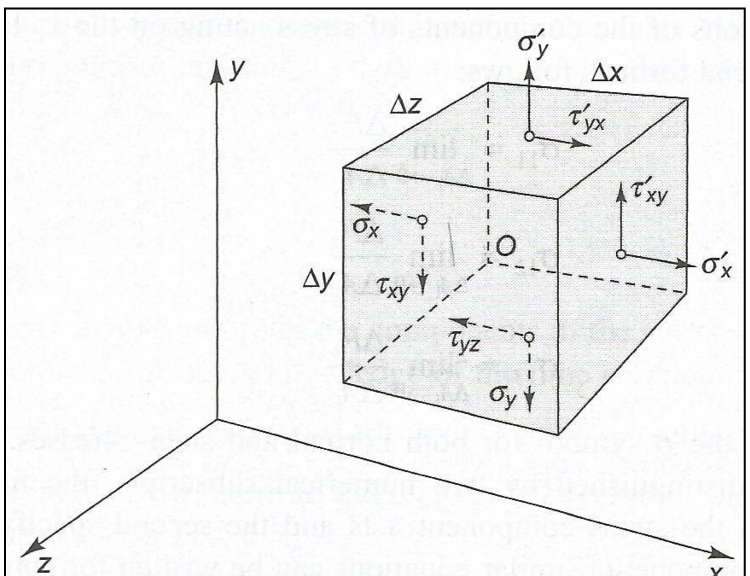


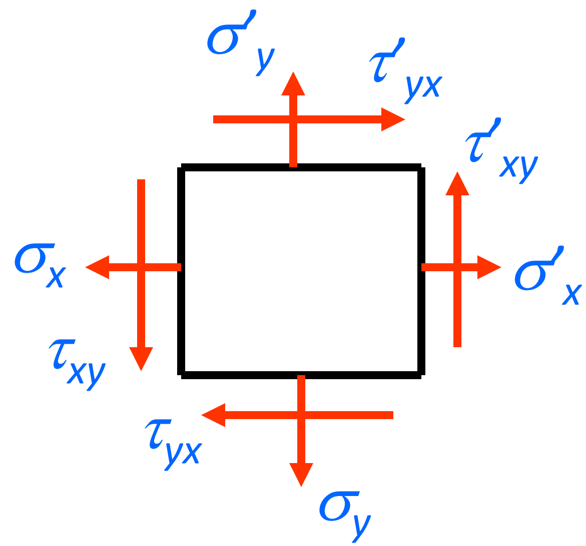
$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

Stress and Strain

Plane stress condition (Stress components in *z* direction are zero)

If take the *xy* plane to be the plane of the sheet, then σ_x , σ'_x , σ_y , σ'_y , τ_{xy} , τ'_{xy} , τ_{yx} and τ'_{yx} will be the only stress components acting on the element, which is under observation.

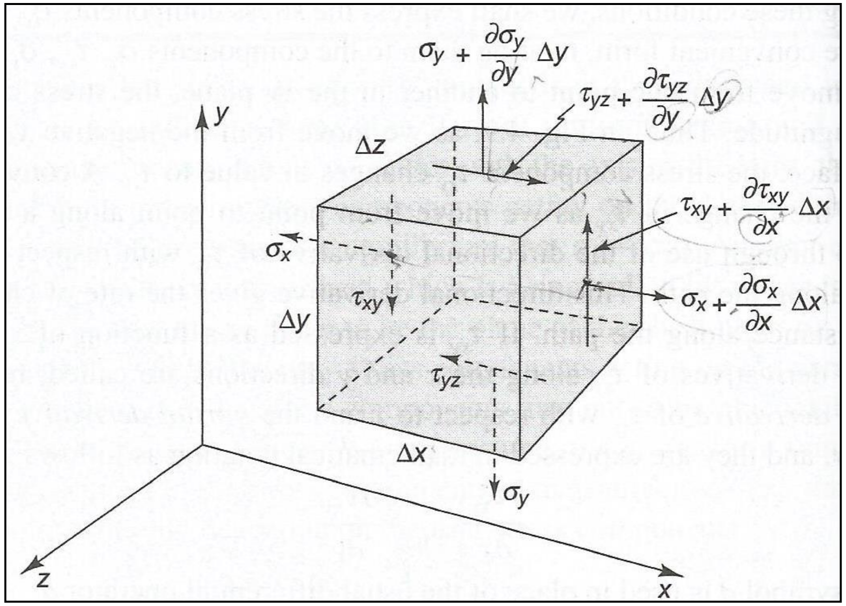




Equilibrium of a Differential Element in Plane Stress

- Stress components in plane stress expressed in terms of partial derivatives.
- Following figure must satisfy equilibrium conditions i.e.

  $\Sigma M = 0$ and $\Sigma F = 0$



Equilibrium of a Differential Element in Plane Stress

ΣM = 0 is satisfied by taking moments about the center of the element

$$\boxed{\sum M = \left\{ \left(\tau_{xy}\Delta y\Delta z\right)\frac{\Delta x}{2} + \left[\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\Delta x\right)\Delta y\Delta z\right]\frac{\Delta x}{2} - \left(\tau_{yx}\Delta x\Delta z\right)\frac{\Delta y}{2} - \left[\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial x}\Delta y\right)\Delta x\Delta z\right]\frac{\Delta y}{2}\right\} k = 0}$$

After simplification, we obtain

$$\boxed{\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\frac{\Delta x}{2} - \tau_{yx} - \frac{\partial \tau_{yx}}{\partial x}\frac{\Delta y}{2} = 0}$$

In the limit as Δx and Δy go to zero $\boxed{\tau_{xy} = \tau_{yx}}$

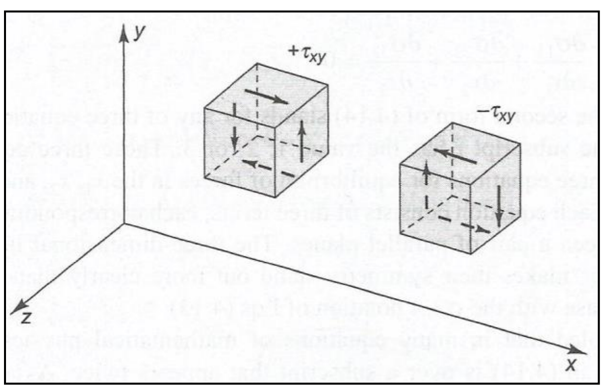
Stress and Strain

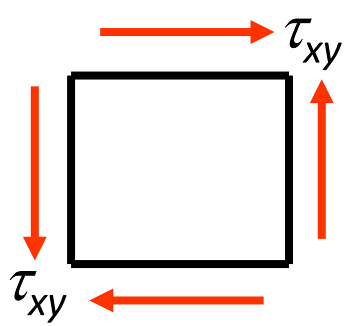
Equality of Cross Shears

This Equation says that in a body in plane stress the shear-stress components on perpendicular faces must be equal in magnitude.

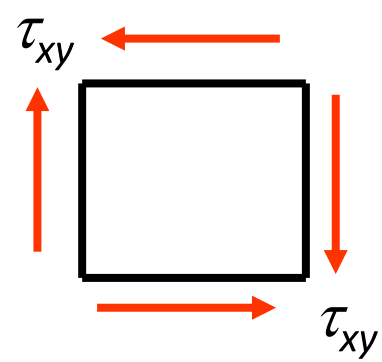
It can also be shown : $\tau_{yz} = \tau_{zy}$ and $\tau_{xz} = \tau_{zx}$

Definition of positive and negative τ_{xy}





Positive Shear



Negative Shear

Equilibrium of a Differential Element in Plane Stress

$\Sigma F = 0$ is satisfied by following two conditions

$$\sum F_x = \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x\right)\Delta y \Delta z + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y\right)\Delta x \Delta z - \sigma_x \Delta y \Delta z - \tau_{yx}\Delta x \Delta z = 0$$

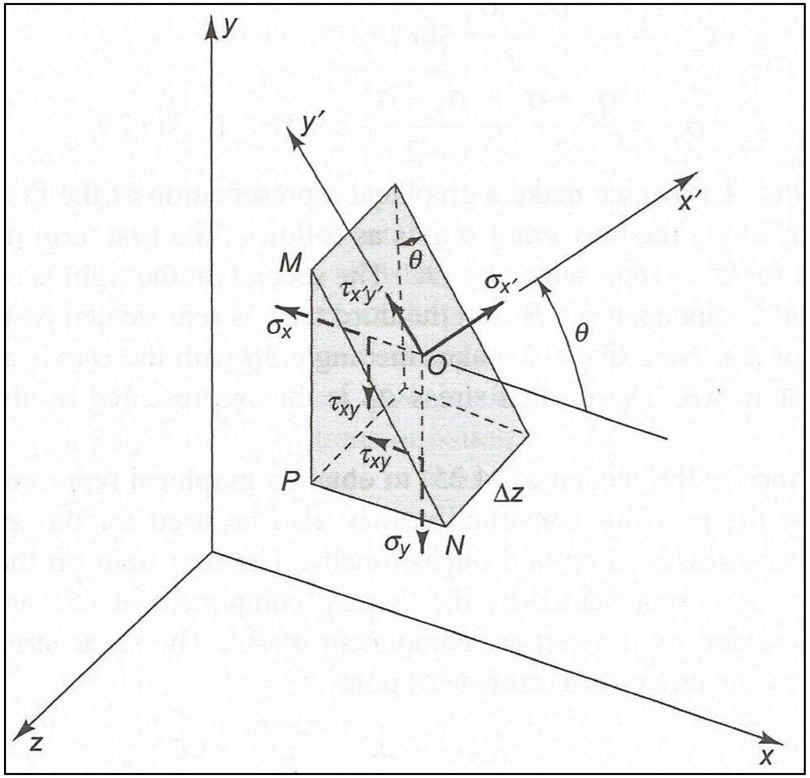
$$\sum F_y = \left(\sigma_y + \frac{\partial \sigma_y}{\partial y}\Delta y\right)\Delta x \Delta z + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\Delta x\right)\Delta y \Delta z - \sigma_y \Delta x \Delta z - \tau_{xy}\Delta y \Delta z = 0$$

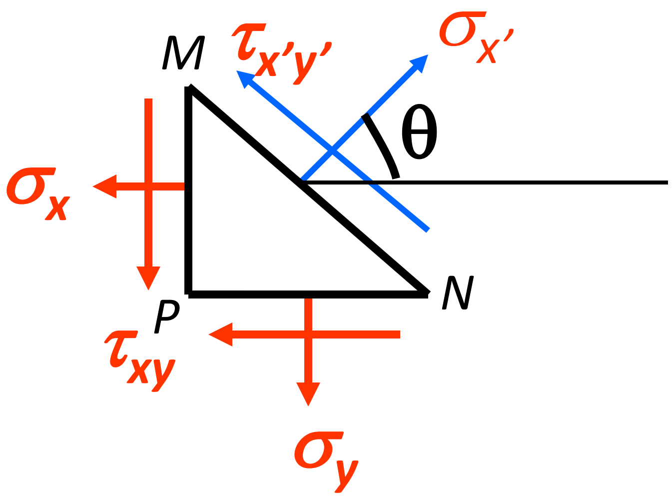
After simplification, we obtain

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

Stress Components Associated with Arbitrarily Oriented Faces in Plane Stress





$MP = MN \cos\theta$

$NP = MN \sin\theta$

Stress and Strain

Resolve forces in normal and along the oblique plane i.e. along *x'* and *y'*

$$\boxed{\sum F_{x'} = \sigma_{x'} MN - \sigma_x MP \cos\theta - \tau_{xy} MP \sin\theta - \sigma_y NP \sin\theta - \tau_{xy} NP \cos\theta = 0}$$

$$\boxed{\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta}$$

$$\boxed{\cos^2\theta = \frac{\cos 2\theta + 1}{2} \quad \text{and} \quad \sin^2\theta = \frac{1 - \cos 2\theta}{2}}$$

$$\boxed{\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta}$$

Stress and Strain

Resolve forces in normal and along the oblique plane i.e. along *x'* and *y'*

$$\boxed{\sum F_{y'} = \tau_{x'y'} MN + \sigma_x MP \sin\theta - \tau_{xy} MP \cos\theta - \sigma_y NP \sin\theta + \tau_{xy} NP \sin\theta = 0}$$

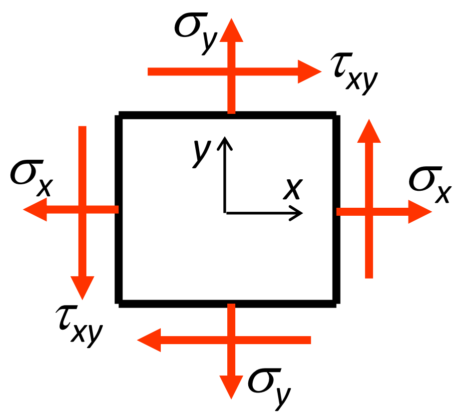
$$\boxed{\tau_{x'y'} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)}$$

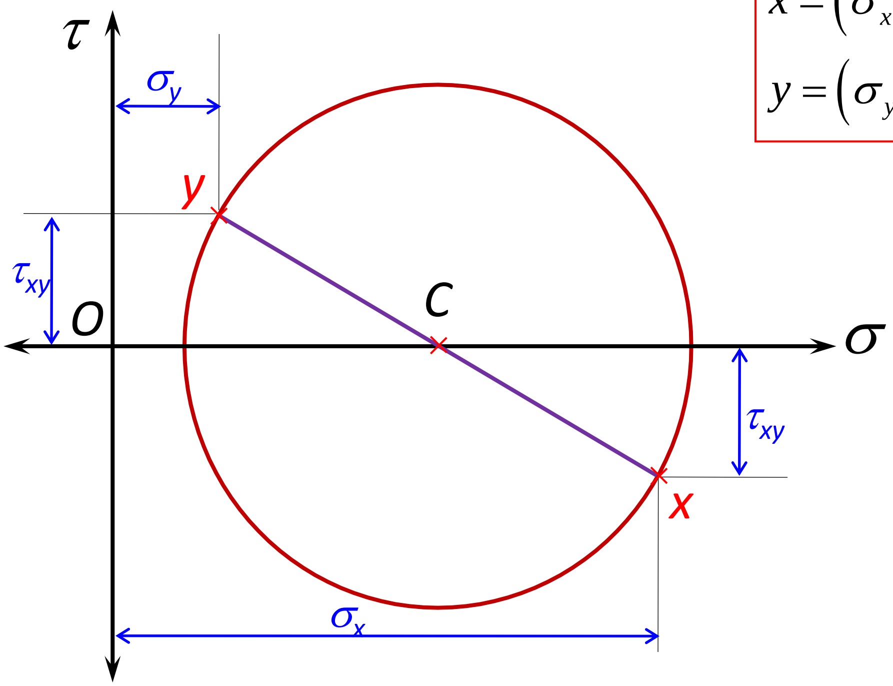
$$\boxed{\tau_{x'y'} = \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta}$$

Similarly

$$\boxed{\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta}$$

Mohr's Circle Representation of Plane Stress





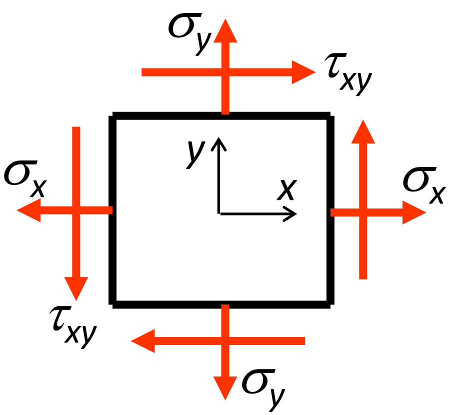
$$x = (\sigma_x, \tau_{xy})$$

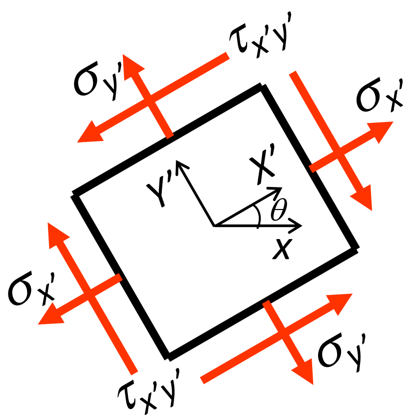
$$y = (\sigma_y, \tau_{xy})$$

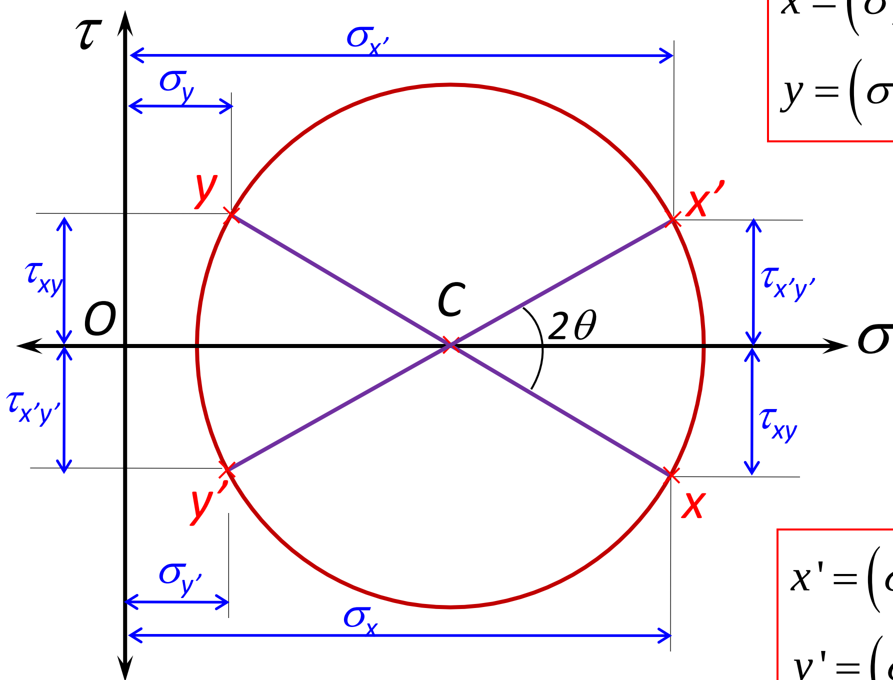
$$R = \left[\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^{\ 2} \right]^{\frac{1}{2}}$$

$$OC = \frac{\sigma_x + \sigma_y}{2}$$

Mohr's Circle Representation of Plane Stress







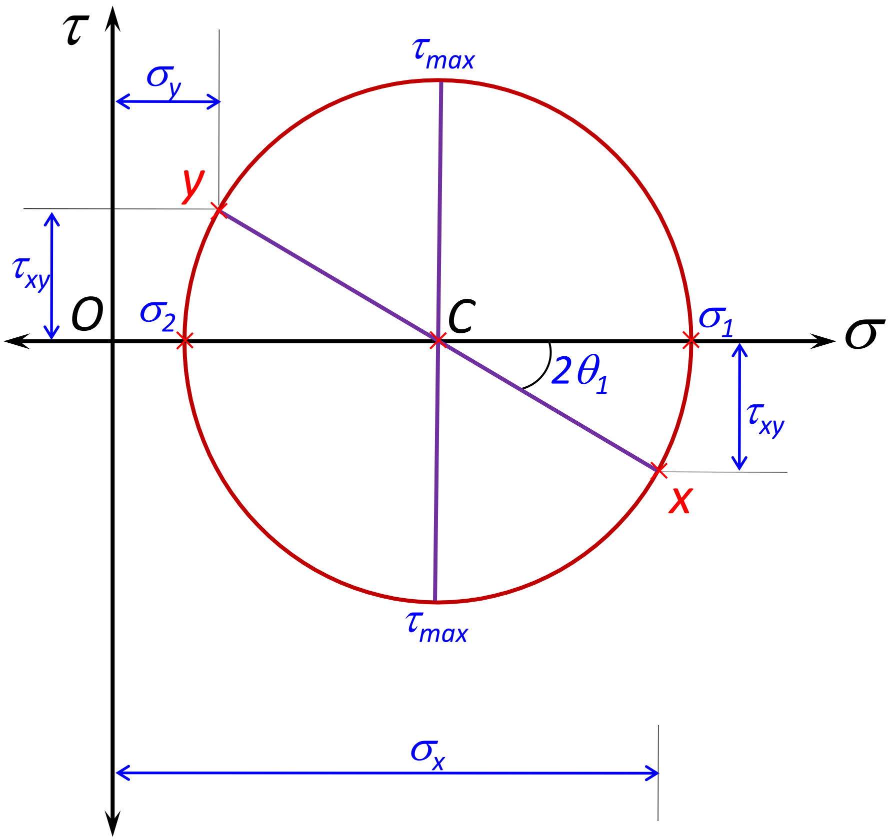
$$x = \left(\sigma_x, \tau_{xy}\right)$$

$$y = \left(\sigma_y, \tau_{xy}\right)$$

$$x' = \left(\sigma_{x'}, \tau_{x'y'}\right)$$

$$y' = \left(\sigma_{y'}, \tau_{x'y'}\right)$$

Mohr's Circle Representation of Plane Stress



$$\tan 2\theta_1 = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

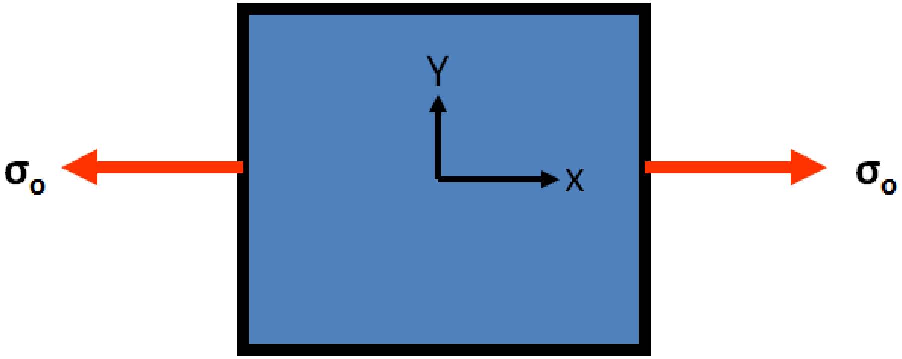
$$\theta_2 = \theta_1 + 90$$

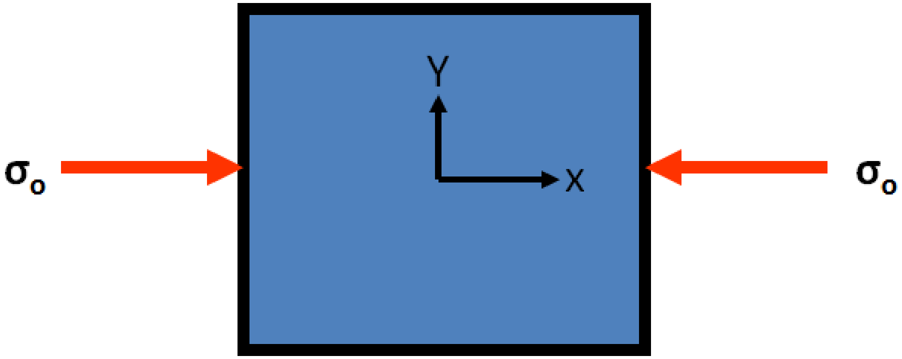
$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\,2}}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\,2}}$$

Problem:

Draw the mohr's circle for following state of stresses





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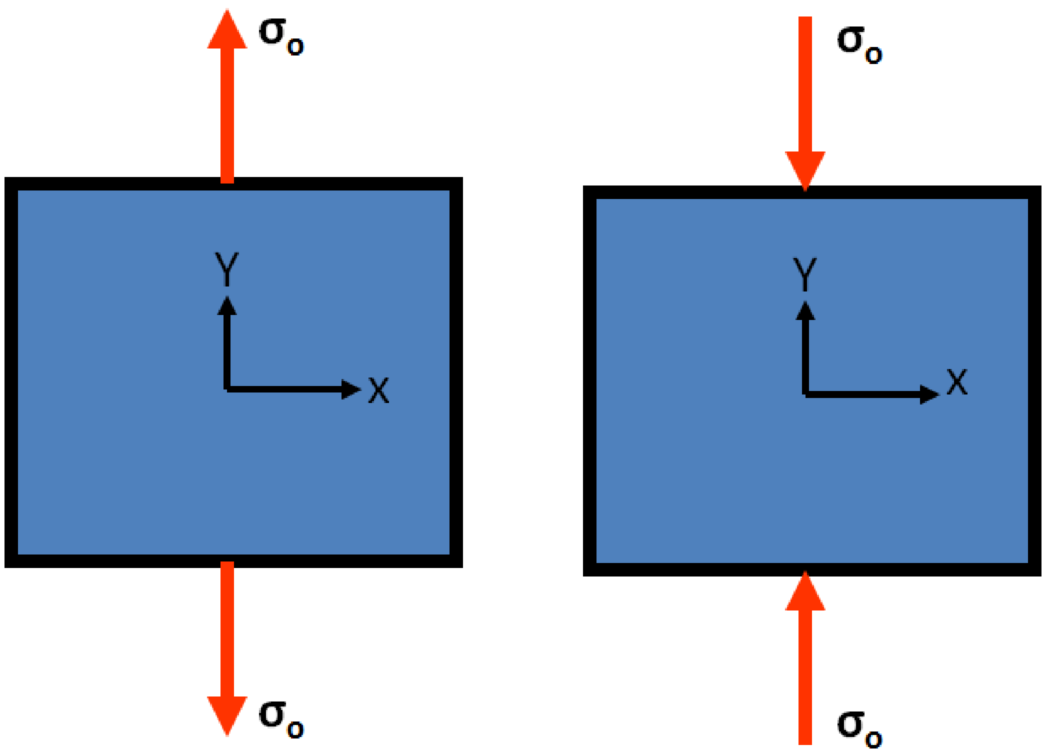

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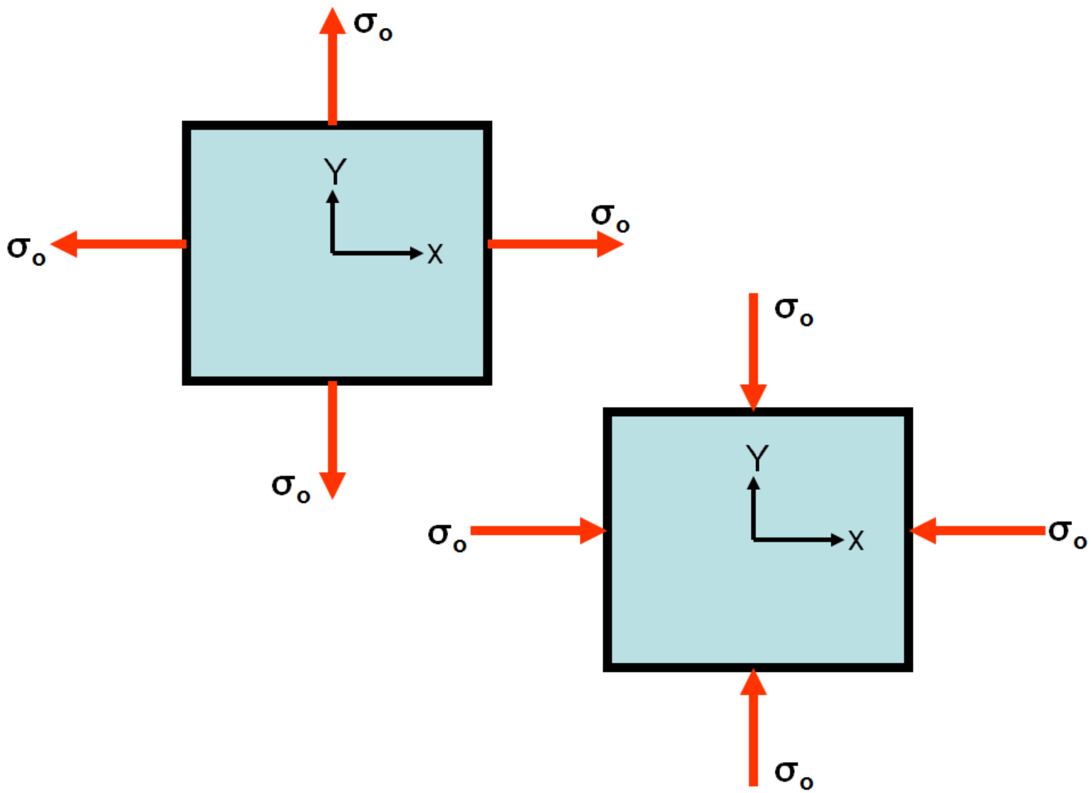

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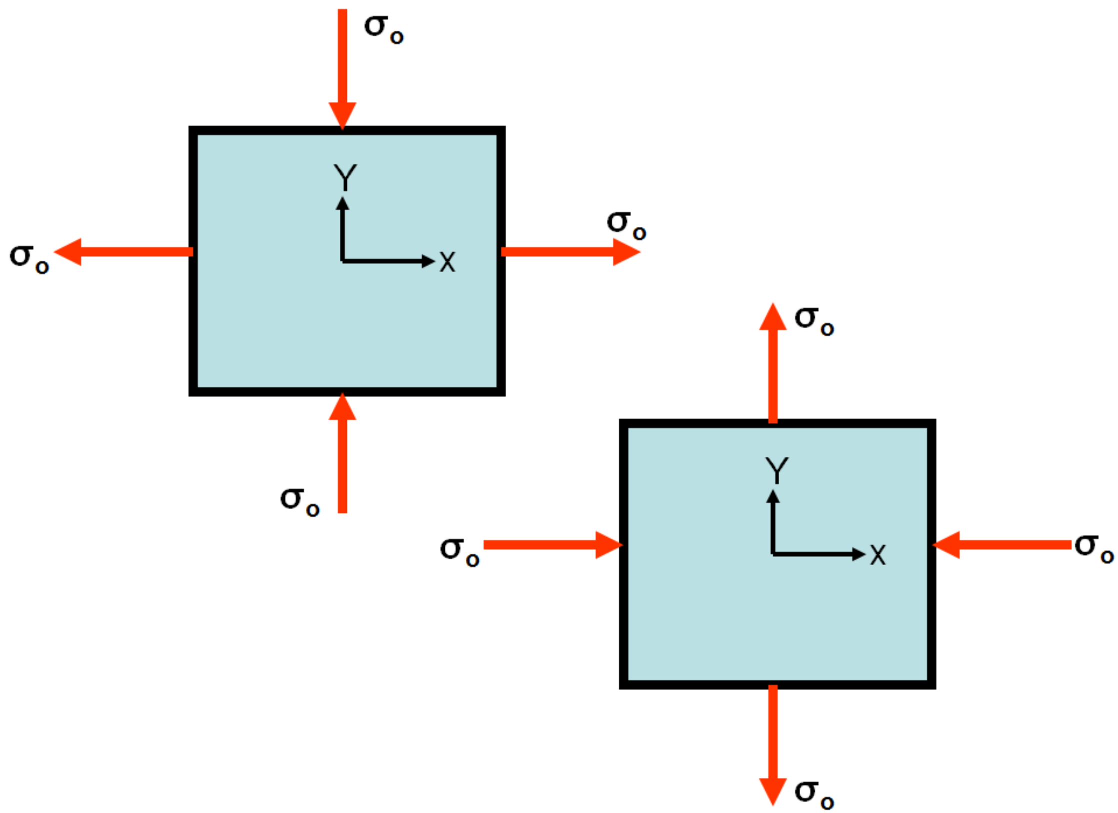

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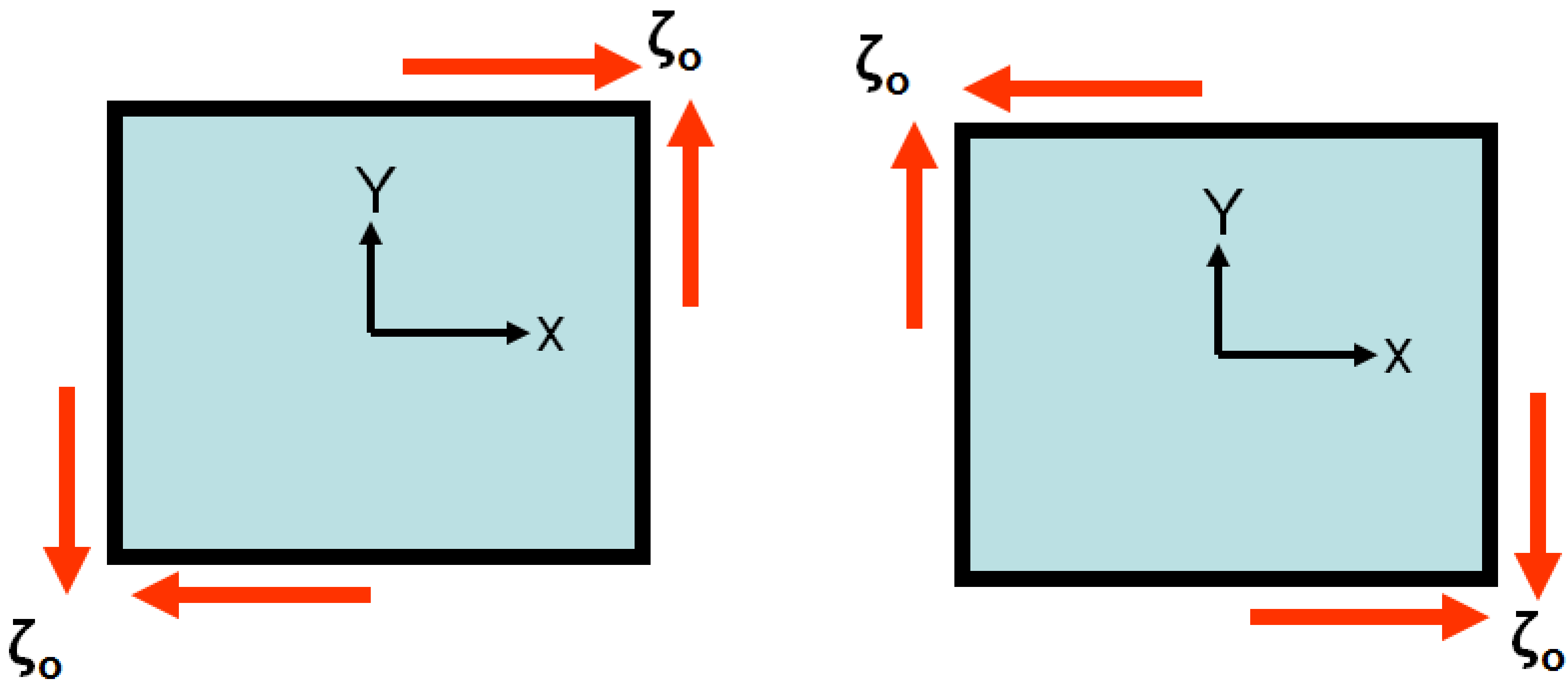
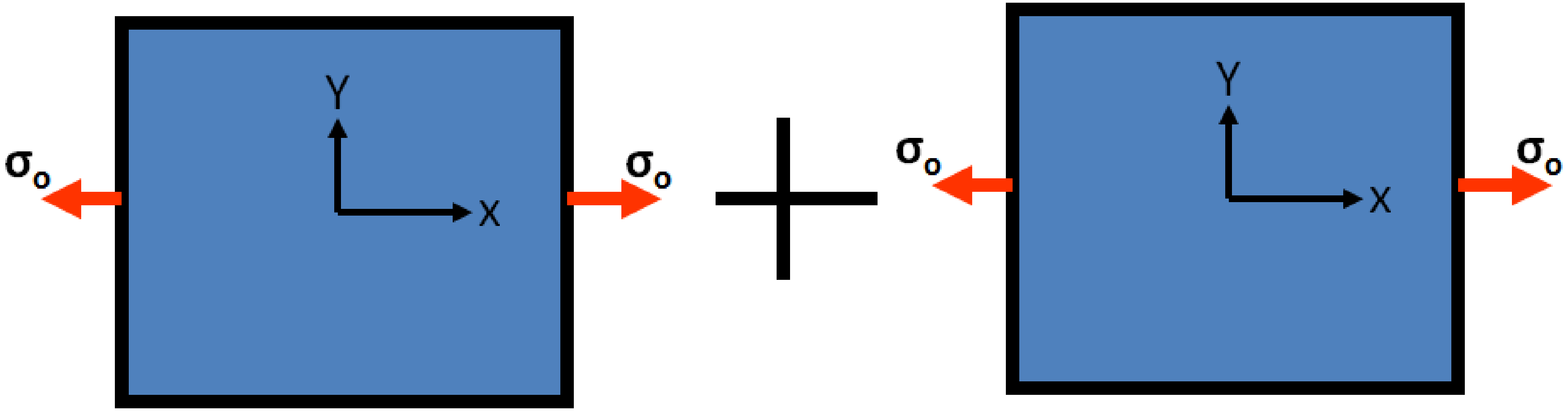


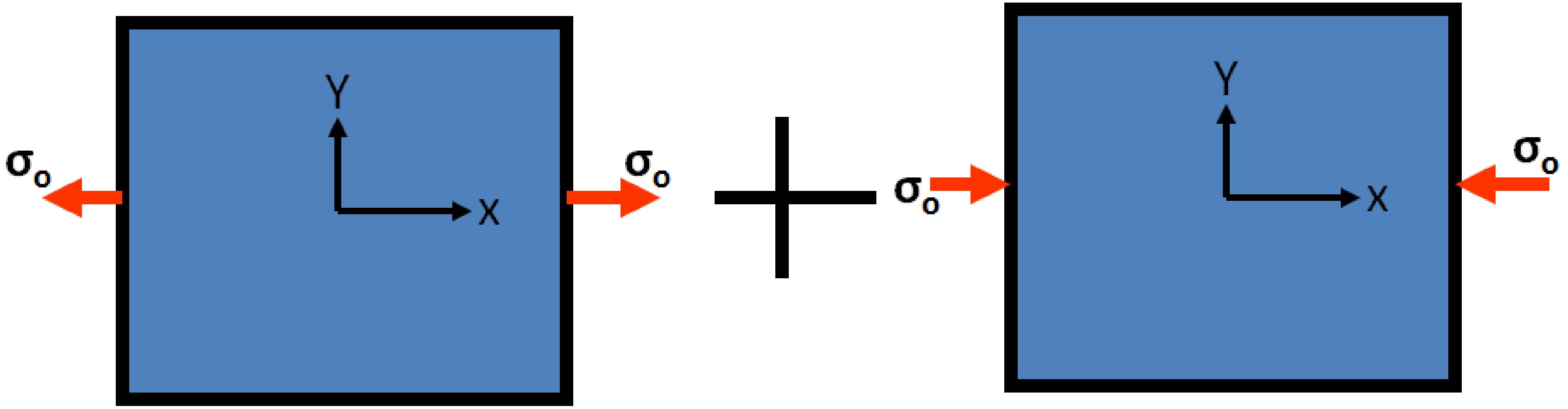
Problem:

Addition of Two States of stress



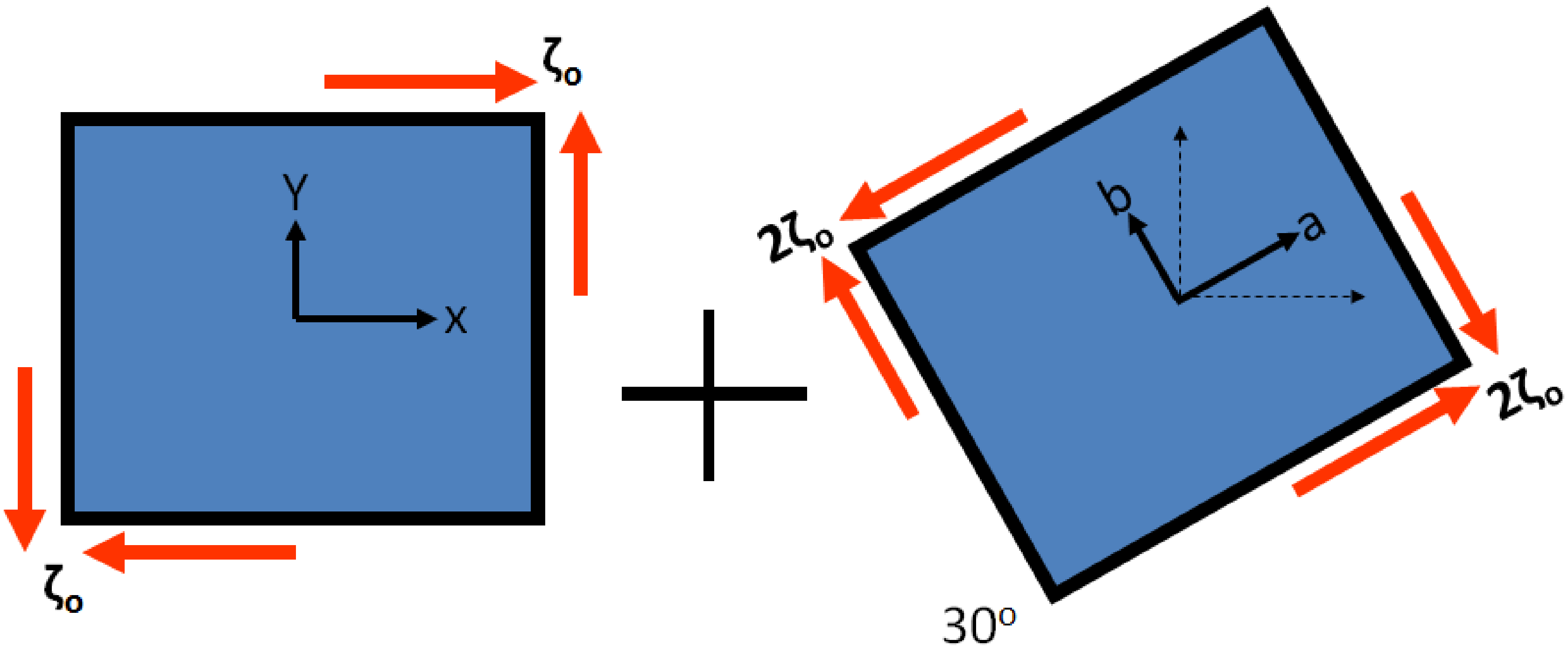
Problem:

Addition of Two States of stress



Problem:

Find the principal stress directions if the stress at a point is sum of the two states of stresses as illustrated



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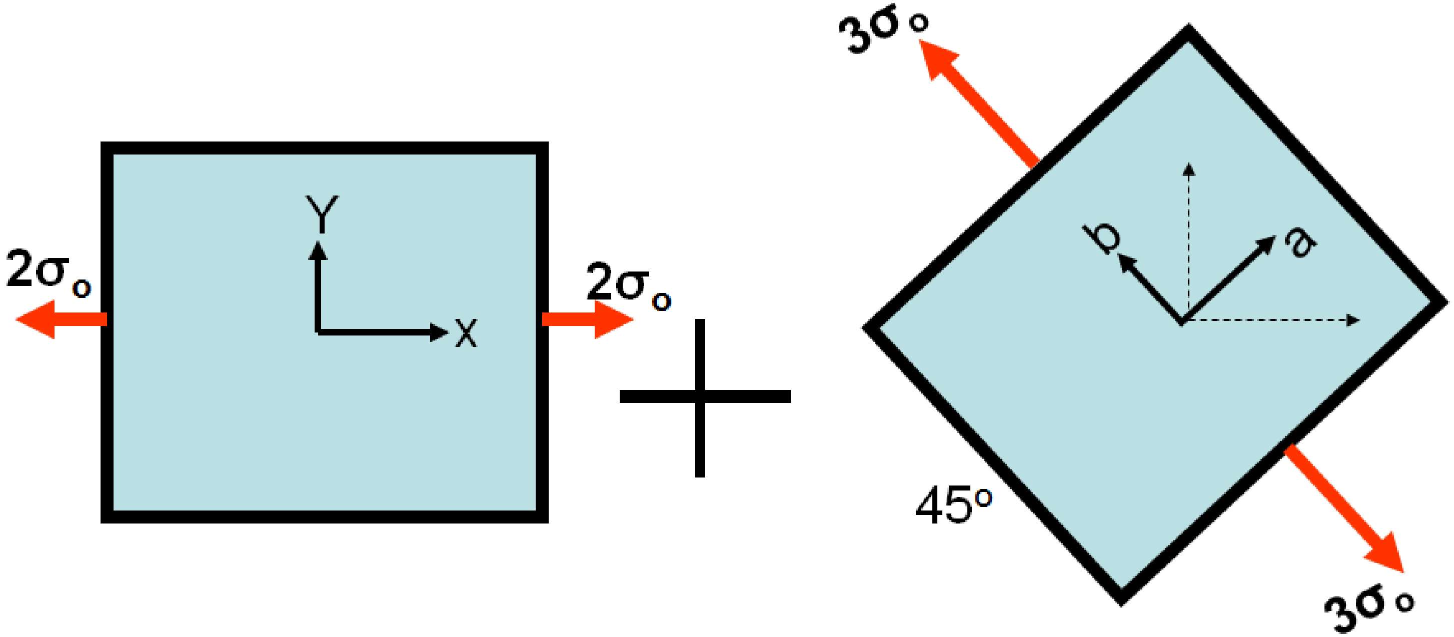


Problem:

Find the principal stress and the orientation of the principal axes of stress for the following cases of plane stress.

a.

$\sigma_x = 40\text{MPa}$

$\sigma_y = 0$

$\tau_{xy} = 80\text{MPa}$

b.

$\sigma_x = 140\text{MPa}$

$\sigma_y = 20\text{MPa}$

$\tau_{xy} = -60\text{MPa}$

c.

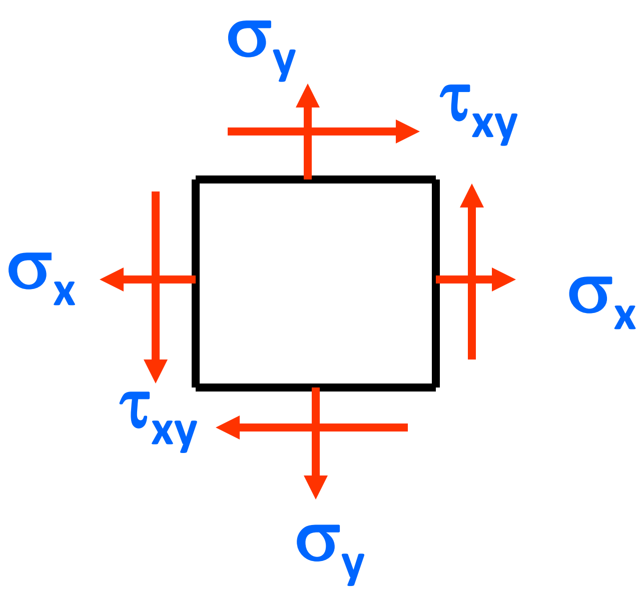
$\sigma_x = -120\text{MPa}$

$\sigma_y = 50\text{MPa}$

$\tau_{xy} = 100\text{MPa}$

Problem:

For given plane stress state find out normal stress and shear stress at a plane 45^0 to x- plane. Also find position of principal planes, principal stresses and maximum shear stress. Draw the mohr's circle and represent all the stresses.



σ_y= 50 MN/m²

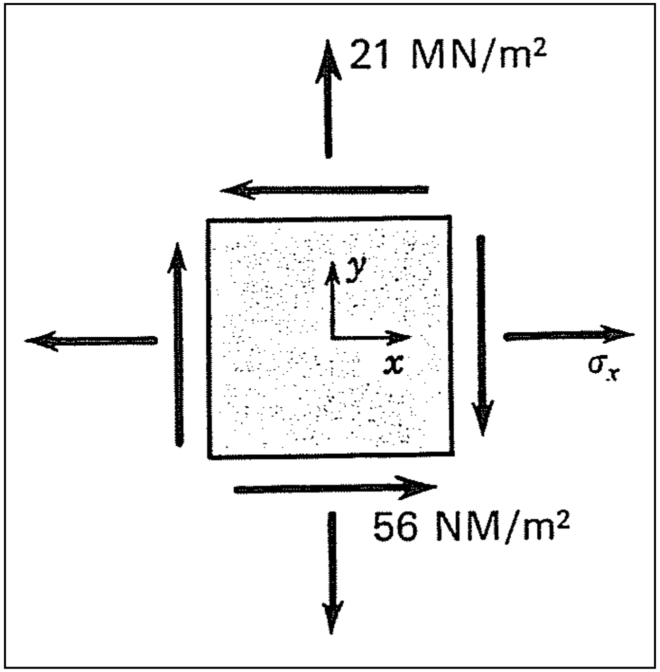
σ_x=110 MN/m²

τ_{xy}= 40 MN/m²

θ= 45^0

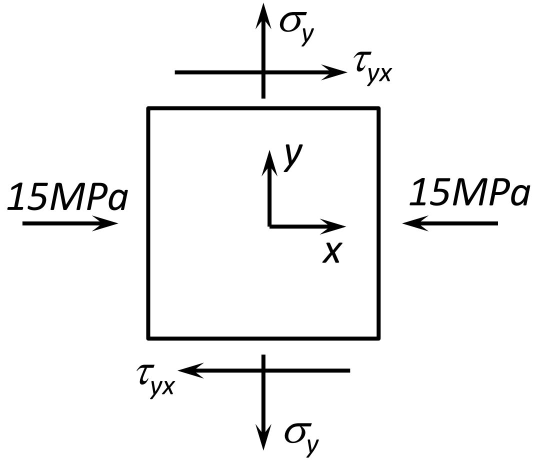
Problem:

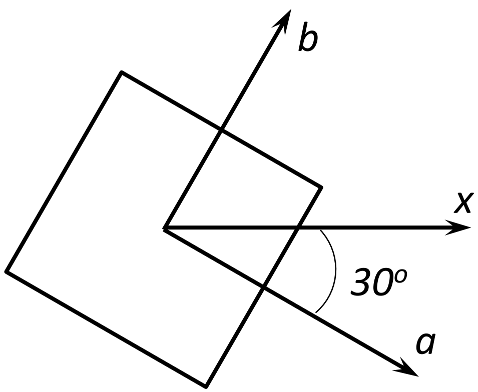
If the minimum principal stress is -7MPa, find σ_x and the angle which the principal axes make with the xy axes for the case of plane stress illustrated



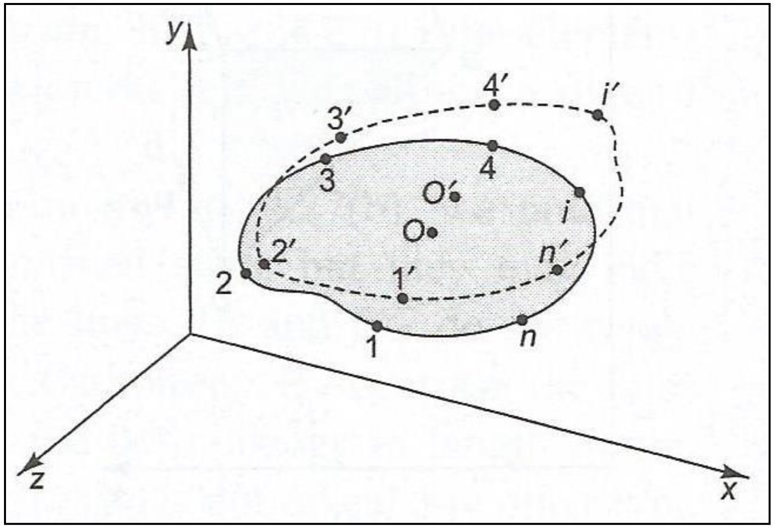
Problem:

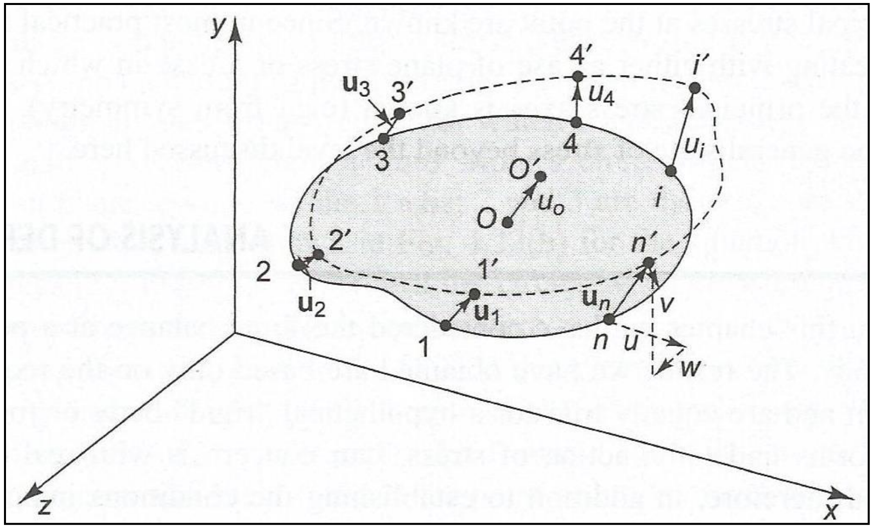
A rectangular plate is under a uniform state of plane stress in the *xy* plane. It is known that the maximum tensile stress acting on any face (whose normal lies in the *xy* plane) is 75MPa. It is also known that on a face perpendicular to the *x* axis there is acting a compressive stress of 15MPa and no shear stress. No explicit information is available as to the values of the normal stress σ_y, and shear stress τ_{xy} acting on the face perpendicular to the *y* axis. Find the stress components acting on the face perpendicular to the *a* and *b* axes which are located as shown in the lower sketch. Report your results in an unambiguous sketch.





Analysis of Deformation





$$u_n = u_n i + u_n j + u_n k$$

Stress and Strain

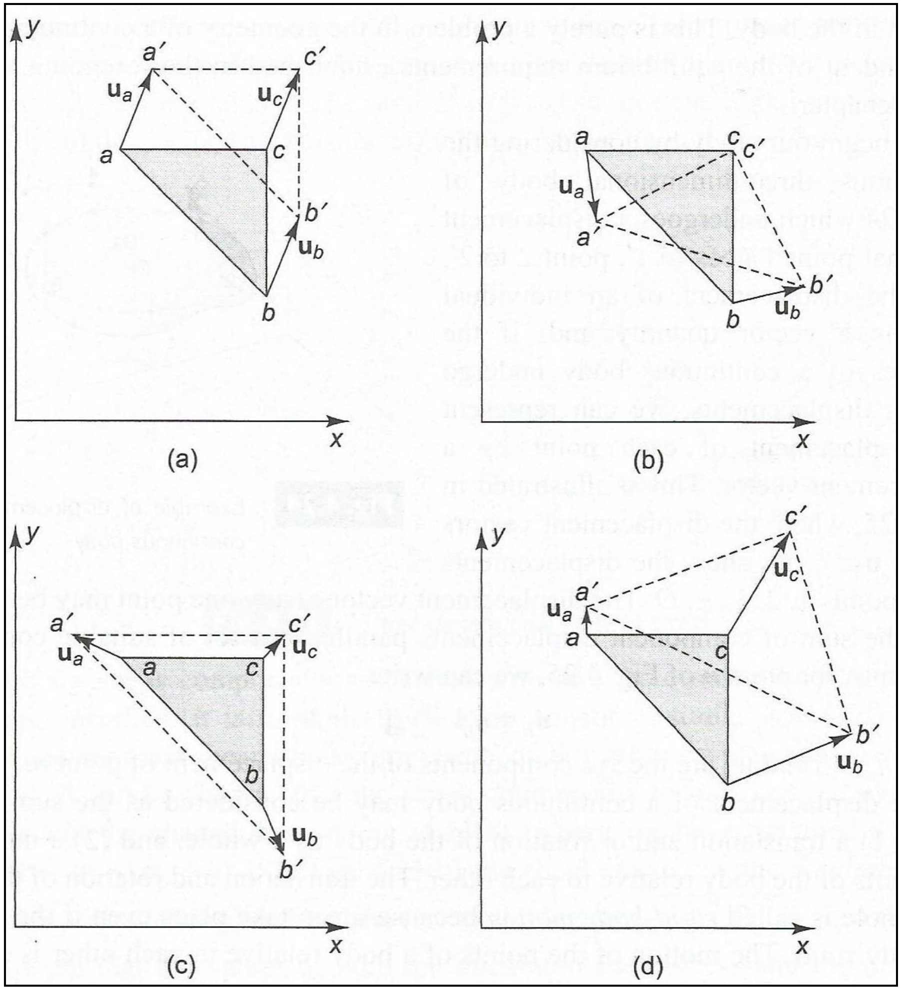
Displacement of Continuous Body

a. Rigid-body translation

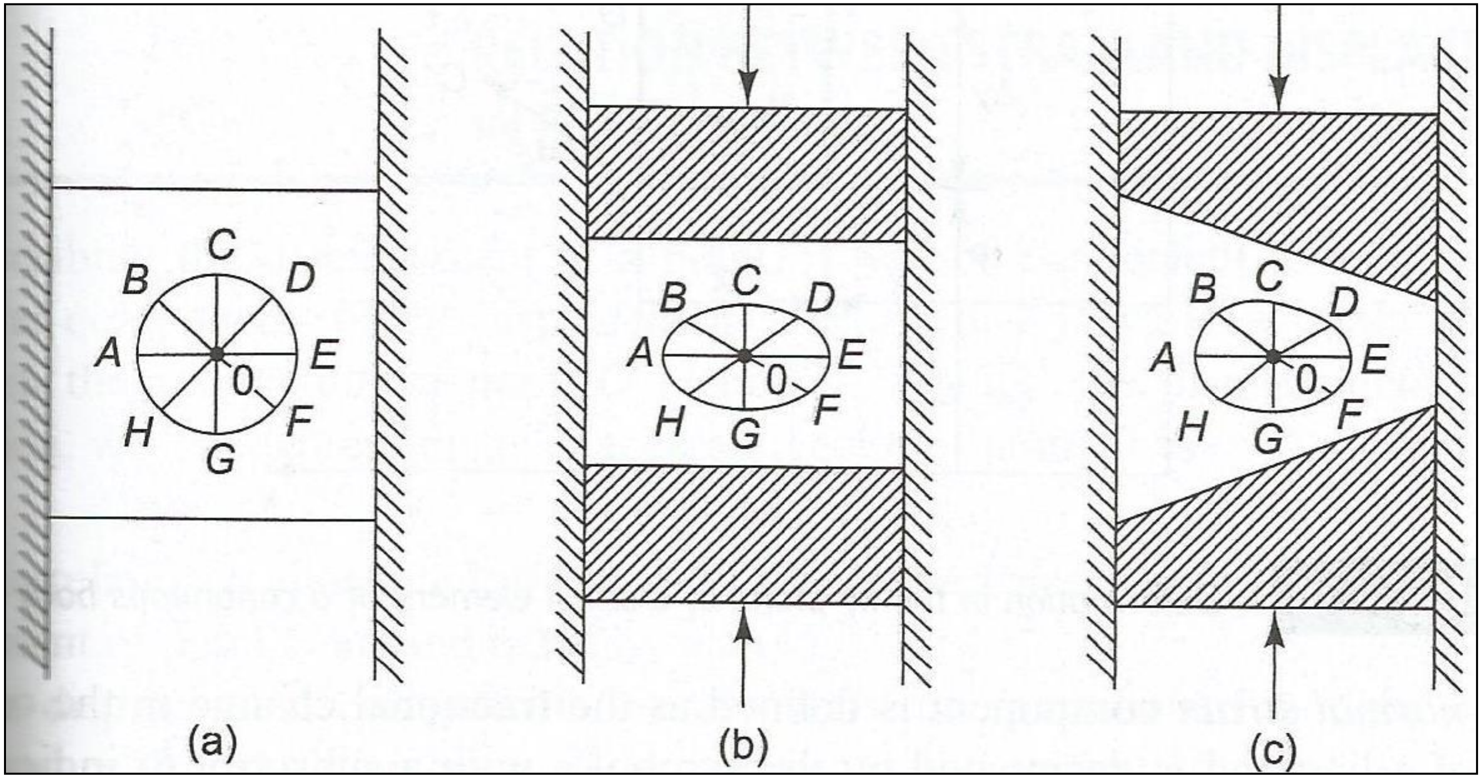
b. Rigid-body rotation about c

c. Deformation without rigid-body motion

d. Sum of all the displacements



Definition of Strain Components

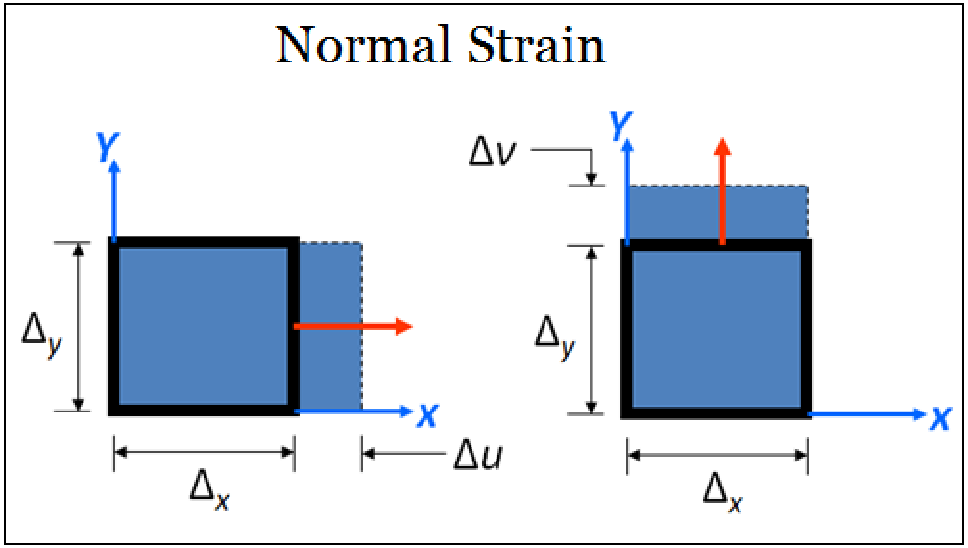


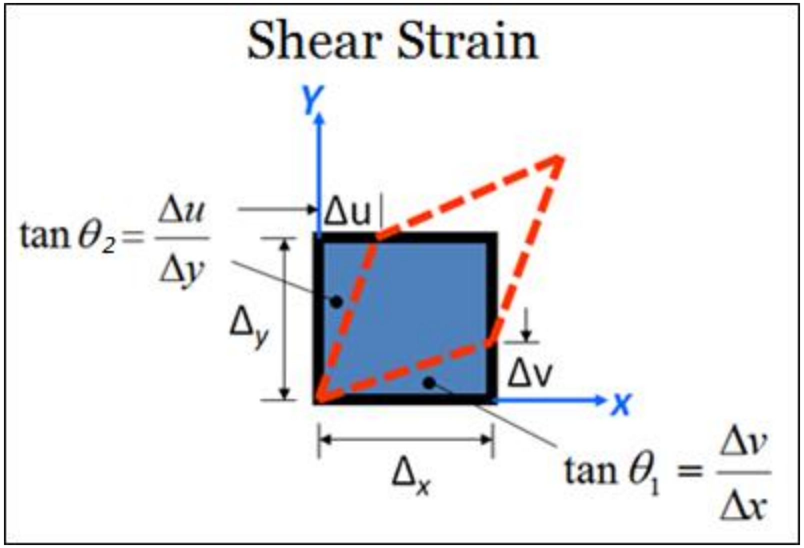
(*b*) Uniform Strain (*c*) Non-uniform Strain

Stress and Strain

Strain

- ❑ Normal Strain
- ❑ Shear Strain





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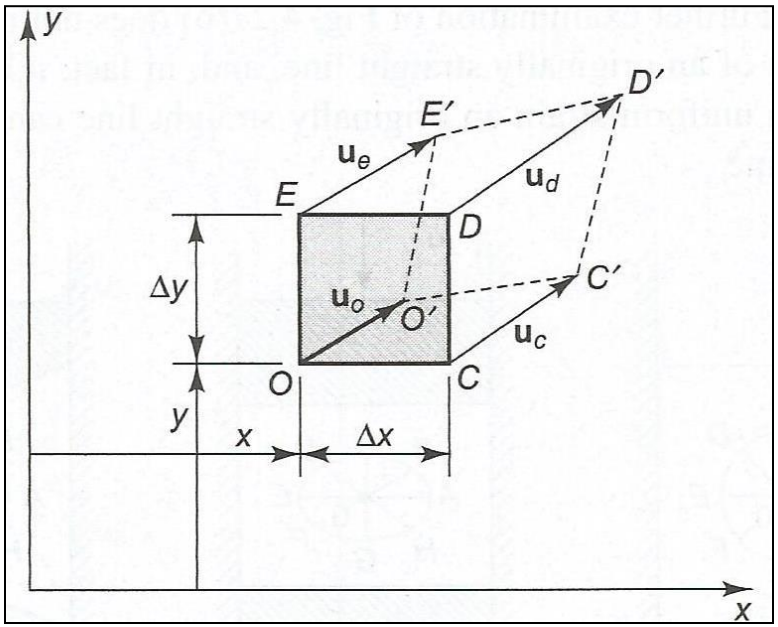
Plane Strain (stain components in *z* direction are zero)



$$\varepsilon_x = \lim_{\Delta x \to 0} = \frac{O'C' - OC}{OC}$$

$$\varepsilon_y = \lim_{\Delta y \to 0} = \frac{O'E' - OE}{OE}$$

$$\gamma_{xy} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\angle COE - \angle C'O'E' \right) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\frac{\pi}{2} - \angle C'O'E' \right)$$

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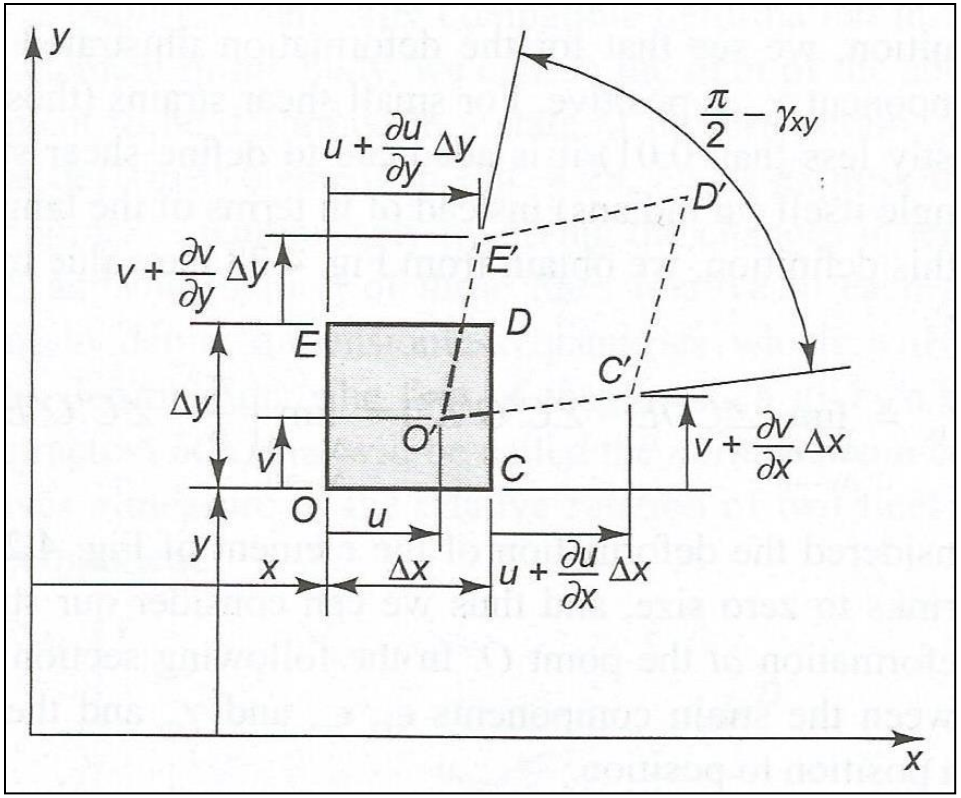
Relation between Strain and Displacement in Plane Strain

- The *x* and *y* components of the displacement of point *O* are indicated by *u* and *v*.
- *u* and *v* must be continuous functions of *x* and *y* to ensure that the displacement be geometrically compatible i.e. no hole or void are created by the displacement.
- Using the concept of partial derivatives, the displacement of point *E* and *C* can be expressed as shown in figure.



Relation between Strain and Displacement in Plane Strain

$$\varepsilon_x = \lim_{\Delta x \to 0} \frac{O'C' - OC}{OC} = \lim_{\Delta x \to 0} \frac{\left[\Delta x + (\partial u/\partial x)\Delta x\right] - \Delta x}{\Delta x} = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \lim_{\Delta y \to 0} \frac{O'E' - OE}{OE} = \lim_{\Delta y \to 0} \frac{\left[\Delta y + (\partial v/\partial y)\Delta y\right] - \Delta y}{\Delta y} = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\frac{\pi}{2} - \angle C'O'E'\right) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left\{\frac{\pi}{2} - \left[\frac{\pi}{2} - \frac{(\partial v/\partial x)\Delta x}{\Delta x} - \frac{(\partial u/\partial y)\Delta y}{\Delta y}\right]\right\} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x}; \qquad \varepsilon_y = \frac{\partial v}{\partial y}; \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}}$$

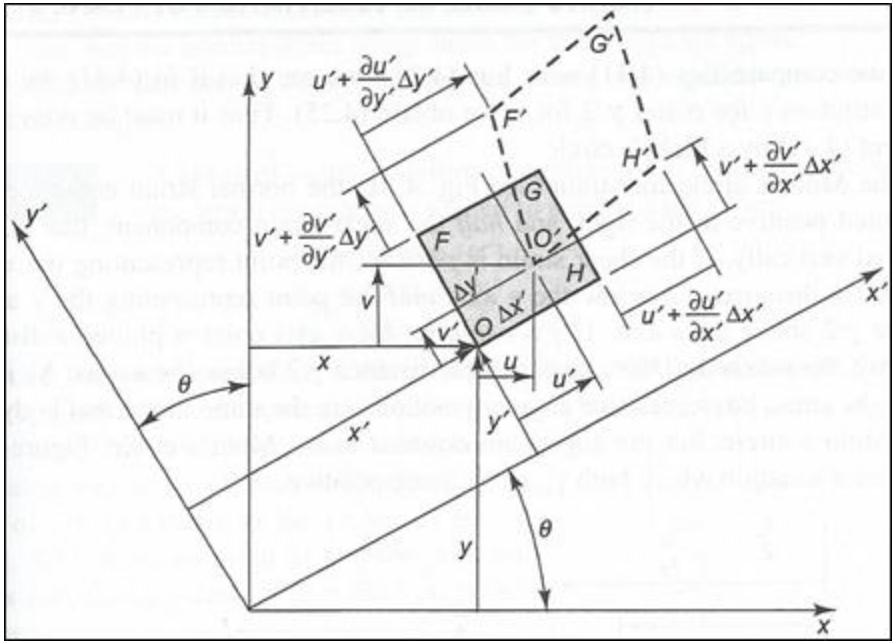
Plain strain condition $\rightarrow$ $\begin{vmatrix} \varepsilon_x & \gamma_{xy} \\ \gamma_{yx} & \varepsilon_y \end{vmatrix}$ where $\boxed{\gamma_{xy} = \gamma_{yx}}$

Stress and Strain

Strain Components Associated with Arbitrary Sets of Axes

$$\varepsilon_{x'} = \frac{\partial u'}{\partial x'}; \qquad \varepsilon_{y'} = \frac{\partial v'}{\partial y'}; \qquad \gamma_{x'y'} = \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}$$

It could be possible to express these displacement components either as functions of *x′* and *y′* or as functions of *x* and *y*.



Strain Components Associated with Arbitrary Sets of Axes

By Chain Rule of Partial Derivatives

$$\varepsilon_{x'} = \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'}$$

$$\varepsilon_{y'} = \frac{\partial v'}{\partial y'} = \frac{\partial v'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial y'}$$

$$\gamma_{x'y'} = \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} = \left(\frac{\partial v'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial x'}\right) + \left(\frac{\partial u'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial y'}\right)$$

From geometry, following relations are obtained

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$
$$u' = u\cos\theta + v\sin\theta$$
$$v' = -u\sin\theta + v\cos\theta$$

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Strain Components Associated with Arbitrary Sets of Axes

$$\varepsilon_{x'} = \left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial v}{\partial x}\sin\theta\right)\cos\theta + \left(\frac{\partial u}{\partial y}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\sin\theta$$

$$\varepsilon_{x'} = \frac{\partial u}{\partial x}\cos^2\theta + \frac{\partial v}{\partial y}\sin^2\theta + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\sin\theta\cos\theta$$

$$\varepsilon_{x'} = \varepsilon_x\cos^2\theta + \varepsilon_y\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$$

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

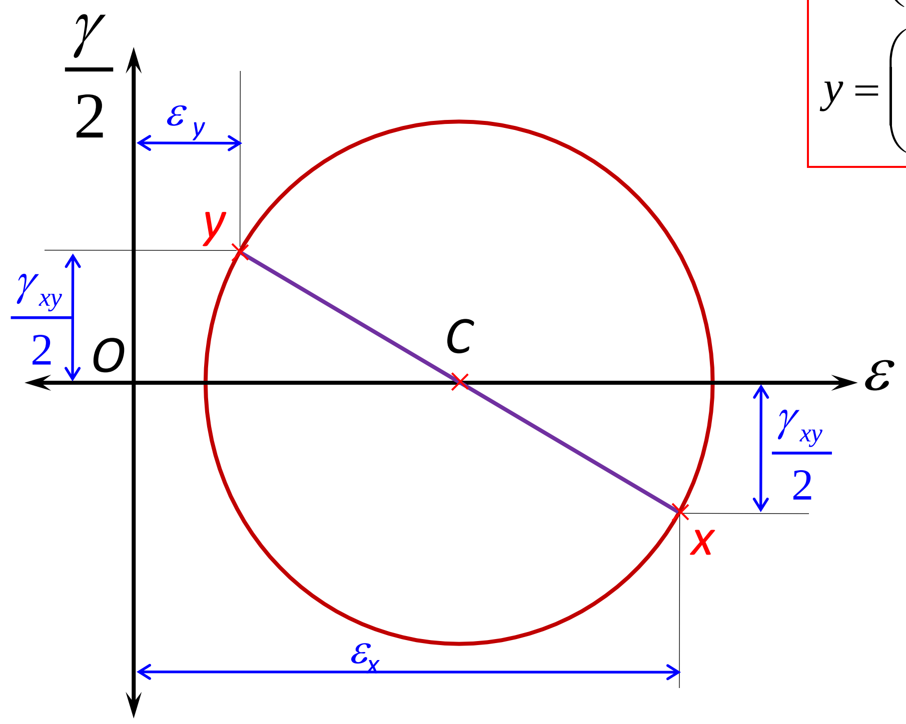
$$\frac{\gamma_{x'y'}}{2} = \frac{\varepsilon_y - \varepsilon_x}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

Mohr's Circle Representation of Plane strain

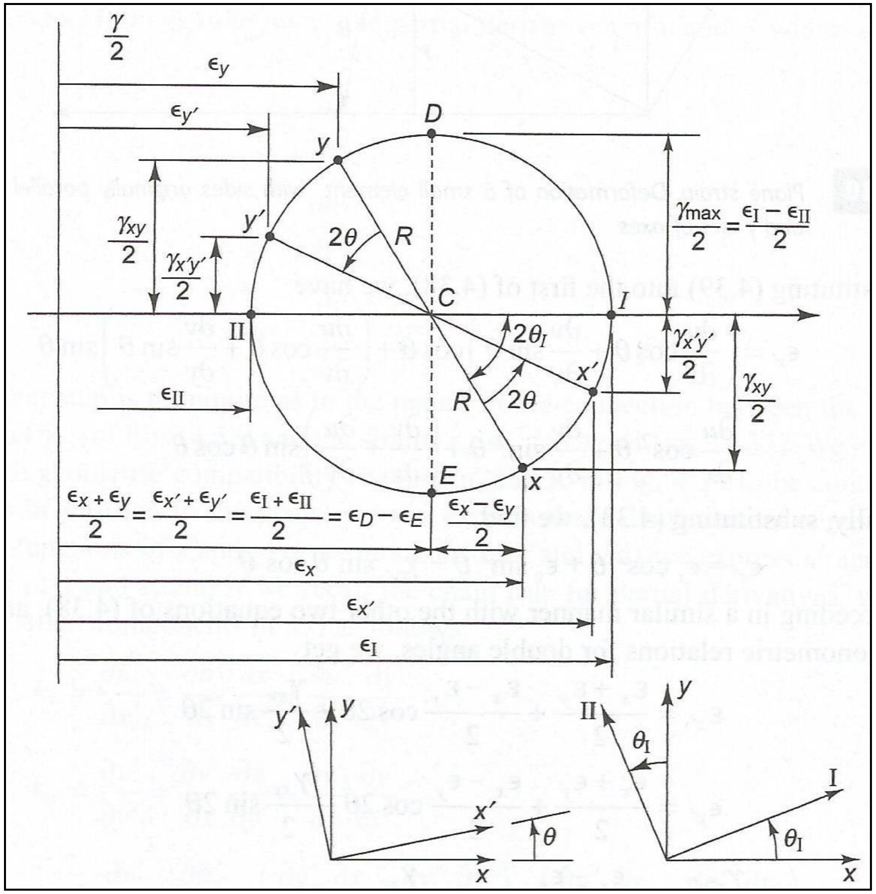
$$\begin{array}{cc} \varepsilon_x & \gamma_{xy} \\ \gamma_{xy} & \varepsilon_y \end{array}$$

$$x = \left(\varepsilon_x, \frac{\gamma_{xy}}{2} \right)$$

$$y = \left(\varepsilon_y, \frac{\gamma_{xy}}{2} \right)$$



Mohr's Circle Representation for any arbitrary Plane



Problem:

A sheet of metal is deformed uniformly in its own plane so that the strain components related to a set of axes *xy* are

$\varepsilon_x = -200 \text{ X } 10^{-6}$

$\varepsilon_y = 1000 \text{X } 10^{-6}$

$\gamma_{xy} = 900 \text{ X } 10^{-6}$

find the strain components associated with a set of axes *x'y'* inclined at an angle of 30° clockwise to the *xy* set. Also to find the principal strains and the direction of the axes on which they exist.

References

1. Introduction to Mechanics of Solids by S. H. Crandall et al (In SI units), McGraw-Hill